With \( g \) in this form, the jump condition holds automatically, and the constant \( \gamma \) is uniquely determined by the requirement that \( \int_a^b g(x,t)\phi(x)dx = 0 \).

**Theorem 4.4**

The modified Green's function for this self-adjoint operator satisfies the two properties:

1. \( g(x,t) = g(t,x) \) (\( g \) is symmetric)

2. \( u = \int_a^b g(x,t)f(t)dt \) satisfies the differential equation \( Lu = f(x) - \phi(x) \int_a^b \phi(t)f(t)dt \) and \( \int_a^b u(x)\phi(x)dx = 0 \), that is, \( u \) is the smallest least squares solution of \( Lu = f \).

Both of these properties are most easily verified by formal arguments. Since \( L \) is self adjoint,

\[
\langle g(x,\xi), Lg(x,\eta) \rangle = \langle Lg(x,\xi), g(x,\eta) \rangle.
\]

Now \( Lg(x,t) = \delta(x-t) - \phi(x)\phi(t) \) so that

\[
g(\eta,\xi) - \phi(\eta) \langle g(x,\xi), \phi(x) \rangle = g(\xi,\eta) - \phi(\xi) \langle g(x,\eta), \phi(x) \rangle,
\]

and since \( \langle g(x,t), \phi(x) \rangle = 0 \), symmetry follows. If \( u = \int_a^b g(x,t)f(t)dt \) then \( Lu = \langle Lg,f \rangle = f(x) - \phi(x) \langle \phi(t), f(t) \rangle \) as promised.

The extension of these statements to include a nonconstant weight function \( \omega(x) \) is direct.

### 4.5 Eigenfunction Expansions for Differential Operators

An operator \( L \) is said to be positive definite if \( \langle Lu, u \rangle > 0 \) for all \( u \neq 0 \). If \( L \) is positive definite then the only solution of \( Lu = 0 \) is the trivial solution \( u = 0 \), since if there is a solution \( u \neq 0 \) with \( Lu = 0 \) then \( \langle Lu, u \rangle = 0 \) contradicts positive-definiteness. If \( L \) is positive definite, \( L \) has a Green's function. (We verified this for separated boundary conditions in Section 4.2.) Further, if \( L \) is self adjoint (with weight function \( \omega(x) = 1 \)), then the Green's function \( g(x,t) \) is symmetric \( g(x,t) = g(t,x) \).

**Definition**

For the differential operator \( L \) (with boundary conditions specified), the pair \( \{\phi, \lambda\} \) is called an eigenvalue, eigenfunction pair if \( L\phi = \lambda\phi, \phi \neq 0 \).
Examples of eigenvalues and eigenfunctions are numerous. For example, the operator $Lu = -\frac{d^2u}{dx^2}$ with boundary data $u(0) = u(1) = 0$ has an infinite set of eigenpairs $\{\phi_n, \lambda_n\} = \{\sin n\pi x, n^2\pi^2\}$.

The most important point of this chapter is now almost trivial to prove.

**Theorem 4.5**

The eigenfunctions of a self-adjoint, invertible second order differential operator form a complete set on $L^2[a, b]$.

**Proof**

Suppose $L$ has a symmetric Green’s function $g(x, t)$. The eigenvalue problem $L\phi = \lambda\phi$ is equivalent to the integral equation

$$\phi = \lambda \int_a^b g(x, t)\phi(t)dt = \lambda K\phi,$$

where $K$ is a symmetric Hilbert-Schmidt operator. The eigenfunctions of $K$ are the same as the eigenfunctions of $L$. However, from Chapter 3, we know that the eigenfunctions of $K$ are complete over the range of $K$. The range of $K$ is the domain of $L$, since $K = L^{-1}$ and the domain of $L$ is the set of $L^2$ functions, with $Lu \in L^2$ which satisfy prescribed boundary conditions. The domain of $L$ is dense in $L^2$, so the eigenfunctions of $L$ are complete in $L^2[a, b]$.

The practical importance of this theorem is that for every self-adjoint second order differential equation, there is a natural coordinate system (i.e., a transform) with which to represent the inverse operator. This coordinate system is the set of mutually orthogonal eigenfunctions which we now know is complete, even though infinite. To solve $Lu = f$ using eigenfunctions, we express $f$ in terms of the eigenfunctions of $L$,

$$f = \sum_{n=1}^{\infty} \alpha_n \phi_n, \quad \alpha_n = \langle f, \phi_n \rangle$$

and seek a solution of the form $u = \sum_{n=1}^{\infty} \beta_n \phi_n$. Since $\{\phi_n\}$ are eigenfunctions, the equation $Lu = f$ transforms to the infinite set of separated algebraic equations

$$\beta_n \lambda_n = \alpha_n$$

which is always solvable and then $u = \sum_{n=1}^{\infty} \frac{(f, \phi_n)\phi_n(x)}{\lambda_n}$ provided $\lambda_n$ is not zero. Another way of expressing this is that, in terms of $\phi_n$, the Green’s function of $L$ is

$$g(x, t) = \omega(t) \sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(t)}{\lambda_n}.$$
Now would be a good time to review the commuting diagrams of transform theory shown in Chapter 1. What we have done here is to solve the equation \( Lu = f \) by first representing \( u \) and \( f \) in the coordinate system of the eigenfunctions of \( L \), solved the diagonalized system of equations in this coordinate representation, and then re-expressed the solution \( u \) in terms of these coordinates. It may happen that \( L \) has a complete set of orthogonal eigenfunctions, but some of the eigenvalues are zero. It follows that the modified Green's function is

\[
g(x, t) = \omega(t) \sum_{\lambda_n \neq 0} \frac{\phi_n(x)\phi_n(t)}{\lambda_n}.
\]

Notice that the eigenfunction expansion based on eigenfunctions of \( L \) will always satisfy homogeneous boundary conditions. If we want to use eigenfunctions to solve a problem with inhomogeneous data, a slightly different approach is necessary.

Suppose we wish to solve \( Lu = f \) subject to inhomogeneous boundary conditions. If \( L \) is self adjoint we observe that

\[
\langle \phi_n, f \rangle = \langle \phi_n, Lu \rangle = \langle L\phi_n, u \rangle - J(\phi_n, u) = \lambda_n \langle \phi_n, u \rangle - J(\phi_n, u)
\]

so that \( \langle \phi_n, u \rangle = \frac{1}{\lambda_n} (\langle \phi_n, f \rangle + J(\phi_n, u)) \) is the algebraic equation which determines the Fourier coefficients of \( u, \langle \phi_n, u \rangle \).

As a specific example, consider the problem \( u'' = f(x) \) subject to inhomogeneous boundary conditions \( u(0) = a, u'(1) = b \). The eigenfunctions for the operator \( Lu = -u'' \) with homogeneous boundary conditions \( u(0) = 0, u'(1) = 0 \) are the trigonometric functions

\[
\phi_n(x) = \sin \left( \frac{2n - 1}{2} \pi x \right), \quad \lambda_n = \left( \left( \frac{2n - 1}{2} \right) \pi \right)^2.
\]

We calculate that

\[
\int_0^1 \phi_n(x)f(x)dx = \int_0^1 \phi_n(x) u''dx = (\phi_n u' - \phi_n' u) \bigg|_0^1 + \int_0^1 \phi_n'' u dx
\]

so that

\[
-\lambda_n \int_0^1 \phi_n u dx = \int_0^1 \phi_n(x)f(x)dx - b\phi_n(1) - a\phi_n'(0).
\]

Since\n
\[
u(x) = \sum_{n=1}^{\infty} a_n \phi_n(x), \quad a_n = \frac{\langle \phi_n, u \rangle}{||\phi_n||^2}
\]

we determine that

\[
-\lambda_n a_n = 2 \int_0^1 \sin \left( \frac{2n - 1}{2} \pi x \right) f(x)dx + 2(-1)^n b - (2n - 1)\pi a.
\]
There is an apparent contradiction in the statement \( u = \sum_{n=1}^{\infty} a_n \phi_n(x) \) satisfies \( u(0) = a, u'(1) = b \), since if one evaluates the infinite sum at \( x = 0 \) or \( x = 1 \) one obtains \( u = 0 \). However, remember that convergence of this series is only in \( L^2 \). It can be shown that \( \lim_{x \to 0^+} u(x) = a \) and \( \lim_{x \to 1^-} u'(x) = b \), and that the function \( u(x) = \sum_{n=1}^{\infty} a_n \phi_n(x) \) differs from a continuous function only at the two points \( x = 0 \) and \( x = 1 \). It is in this \( L^2 \) sense that \( u \) satisfies the inhomogeneous boundary conditions.

The most important group of eigenfunctions occur as eigenfunctions of Sturm-Liouville operators. Suppose \( Lu = -(p(x)u')' + q(x)u \) where \( p(x) > 0 \) and \( q(x) > 0 \) except possibly on a set of measure zero of the interval \( x \in [a, b] \). We can easily check that \( L \) is positive definite since

\[
\langle Lu, u \rangle = -pu'|^b_a + \int_a^b (p(x)u''(x) + q(x)u^2(x)) \, dx
\]

is positive for nonzero \( u \), provided the boundary conditions are such that \( pu'|^b_a \leq 0 \). If the operator \( L \) is of the form \( Lu = \frac{1}{\omega(x)}(-(p(x)u')' + q(x)u) \) then the appropriate inner product has weight function \( \omega(x) \), with no other changes.

We can prove completeness of eigenfunctions for certain operators which are not positive definite. For example, suppose the Sturm-Liouville operator \( Lu = -(p(x)u')' + q(x)u \), \( p(x) > 0 \), is not positive definite although \( q(x) \) is bounded below. If \( q(x) > -\lambda_0 \), then \( Lu = -(p(x)u')' + q(x)u + \lambda_0 u \) is a positive definite operator, and although the eigenvalues of \( L \) are all shifted, the eigenfunctions of \( L \) are unchanged and hence complete. For example, the operator \( Lu = -u'' \), with boundary conditions \( u'(0) = u'(1) = 0 \), is not positive definite, although the shifted operator \( Lu = -u'' + u \) with the same boundary conditions is. As a result the eigenfunctions of this operator \( \{\cos n\pi x\}_{n=0}^{\infty} \) are complete on \( L^2[0, 1] \).

The remainder of this chapter gives a sample of the eigenfunctions that result from Sturm-Liouville operators.

### 4.5.1 Trigonometric Functions

The trigonometric functions are easily shown to be complete using this theory. The differential operator \( Lu = -\frac{d^2u}{dx^2} \) with boundary conditions \( u(0) = u(1) = 0 \) has eigenfunctions \( \{\sin n\pi x\}_{n=1}^{\infty} \) which are complete on \( L^2[0, 1] \). Completeness of the full trigonometric series follows by examining the operator \( Lu = -\frac{d^2u}{dx^2} + u \) subject to periodic conditions \( u(0) = u(1) \) and \( u'(0) = u'(1) \) for which the eigenfunctions are \( \{\sin 2n\pi x\}_{n=1}^{\infty} \) and \( \{\cos 2n\pi x\}_{n=0}^{\infty} \). Note that the eigenvalues with \( n \neq 0 \) have geometric multiplicity 2.

As we noted in Chapter 2, the completeness of Fourier series only guarantees convergence in the sense of \( L^2 \). Pointwise convergence is another matter entirely.
7.2 Fourier Transforms

7.2.1 Transform Pairs

There is an interesting relationship between Green’s functions, eigenvalue expansions and contour integration in the complex plane. Suppose we have an operator $L$ with a complete set of orthonormal eigenfunctions $\{\phi_k\}$ where $L\phi_k = \lambda_k \phi_k$. For every $u$ in the Hilbert space, $u = \sum_{k=1}^{\infty} \alpha_k \phi_k$ where $\alpha_k = \langle u, \phi_k \rangle$. For this $u$, $Lu = \sum_{k=1}^{\infty} \alpha_k \lambda_k \phi_k$, if $Lu$ makes sense. Furthermore, $L^2 u = L(Lu) = \sum_{k=1}^{\infty} \alpha_k \lambda_k^2 \phi_k$ and obviously, $L^{(n)} u = \sum_{k=1}^{\infty} \alpha_k \lambda_k^n \phi_k$, again, provided these all make sense. If $f(x)$ is a polynomial, one can define the operator $f(L)$ by $f(L)u = \sum_{k=1}^{\infty} \alpha_k f(\lambda_k) \phi_k$ and finally, if $f$ is an analytic function, the same definition makes good sense, namely

$$f(L)u = \sum_{k=1}^{\infty} \alpha_k f(\lambda_k) \phi_k.$$

If we wish to invert the operator $L - \lambda$ for some complex number $\lambda$, we can identify $f(x) = \frac{1}{x-\lambda}$ and the operator $(L - \lambda)^{-1} = f(L)$ is, by our previous definition,

$$(L - \lambda)^{-1} u = \sum_{k=1}^{\infty} \frac{\alpha_k \phi_k}{\lambda_k - \lambda}.$$

Of course, we already know that, if $L$ is a differential operator, the inverse operator $(L - \lambda)^{-1}$ is represented by the Green’s function, say

$$(L - \lambda)^{-1} u = \int G(x, \xi; \lambda) u(\xi) d\xi$$

where $\lambda$ is treated as a parameter, and integration is taken over the appropriate spatial domain. Notice that this implies that

$$- \int G(x, \xi; \lambda) u(\xi) d\xi = \sum_{k=1}^{\infty} \frac{\alpha_k \phi_k}{\lambda - \lambda_k}.$$

If both of these expressions are defined for $\lambda$ in the complex plane, we can integrate counterclockwise with respect to $\lambda$ around a large circle, and let $R$, the radius of the circle, go to infinity. The result is

$$\int_{C_\infty} \left( \int G(x, \xi; \lambda) u(\xi) d\xi \right) = - \int_{C_\infty} \sum_{k=1}^{\infty} \frac{\alpha_k \phi_k}{\lambda - \lambda_k} d\lambda$$

$$= -2\pi i \sum_{k=1}^{\infty} \alpha_k \phi_k = -2\pi i u(x),$$
because of the residue theorem. Apparently,

\[ u(x) = -\frac{1}{2\pi i} \int_{C_\infty} \left( \int G(x, \xi; \lambda) u(\xi) d\xi \right) d\lambda. \]

Here we have an interesting statement. Namely, there is some function which when integrated against reasonable functions \( u(\xi) \) reproduces \( u(x) \). We know from the theory of distributions that this cannot occur for any honest function, but it must be that

\[ -\frac{1}{2\pi i} \int_{C_\infty} G(x, \xi; \lambda) d\lambda = \delta(x - \xi) \]

interpreted in the sense of distribution.

This rather curious formula has a most powerful interpretation, namely it shows how to get representations of the "\( \delta \)-function" by knowing only the Green's function for the differential operator of interest. As we shall see, this generates directly the transform pair appropriate to the operator \( L \).

To see how this formula works to generate transform pairs, we examine the familiar operator \( Lu = -u'' \) with boundary conditions \( u(0) = u(1) = 0 \). We calculate the Green's function for \( (L - \lambda)u \) to be

\[
G(x, \xi; \lambda) = \begin{cases} 
\frac{\sin \sqrt{\lambda} x}{\sqrt{\lambda}} \frac{\sin \sqrt{\lambda}(1 - \xi)}{\sin \sqrt{\lambda}}, & 0 \leq x < \xi \leq 1, \\
\frac{\sin \sqrt{\lambda} \xi}{\sqrt{\lambda}} \frac{\sin \sqrt{\lambda}(1 - x)}{\sin \sqrt{\lambda}}, & 0 \leq \xi < x \leq 1.
\end{cases}
\]

To evaluate \( \int_{C_\infty} G(x, \xi; \lambda) d\lambda \), observe that there are simple poles at \( \lambda = n^2 \pi^2 \) for \( n = 1, 2, \ldots \). The point \( \lambda = 0 \) is not a pole or branch point since for small \( \lambda \),

\[
G(x, \xi; \lambda) \sim \begin{cases} 
x(1 - \xi) & 0 \leq x < \xi \leq 1 \\
\xi(1 - x) & 0 \leq \xi < x \leq 1.
\end{cases}
\]

Applying the residue theorem, we obtain that

\[ \delta(x - \xi) = -\frac{1}{2\pi i} \int_{C_\infty} G(x, \xi; \lambda) d\lambda = 2 \sum_{k=1}^{\infty} \sin k\pi x \sin k\pi \xi. \]

This representation of the delta function is equivalent to the sine Fourier series since for any smooth \( f \) with compact support in the interval \([0, 1]\),

\[
f(x) = \int_{0}^{1} \delta(x - \xi) f(\xi) d\xi.
\]
\[ = \sum_{k=1}^{\infty} \sin k\pi x \left( 2 \int_{0}^{x} f(\xi) \sin k\pi \xi d\xi \right) \]
\[ = \sum_{k=1}^{\infty} \alpha_k \sin k\pi x \]

where \( \alpha_k = 2 \int_{0}^{1} f(\xi) \sin k\pi \xi d\xi \).

It is a straightforward calculation to extend the representation of the delta function to Sturm-Liouville operators
\[ Lu - \lambda u = - \left( \frac{1}{\omega}(pu')' + qu + \lambda u \right). \]

We assume that the boundary conditions correspond to a self-adjoint operator, and that the eigenfunctions satisfy
\[ L\phi_k - \lambda_k \phi_k = 0 \]

where \( Lu = -\frac{1}{\omega}(pu')' - qu \), and are complete and orthonormal with respect to the inner product with weight function \( \omega > 0 \). Since the functions \( \phi_k \) are complete, we can represent the Green's function as
\[ G(x, \xi; \lambda) = \sum_{k=1}^{\infty} \alpha_k(\xi) \phi_k(x), \]

where \( \alpha_k(\xi) \) are the Fourier coefficients \( \alpha_k = \langle \phi_k, G \rangle \). Since \( G \) is defined by
\[ LG - \lambda G = \delta(x - \xi) \]
then
\[ \alpha_k(\xi) = \langle \phi_k, G \rangle = \frac{\phi_k(\xi) \omega(\xi)}{\lambda_k - \lambda} \]
so that
\[ G(x, \xi; \lambda) = \sum_{k=1}^{\infty} \frac{\phi_k(\xi) \phi_k(x) \omega(\xi)}{\lambda_k - \lambda}. \]

It follows that
\[ \frac{1}{2\pi i} \int_{C_\infty} G(x, \xi; \lambda) d\lambda = -\sum_{k=1}^{\infty} \phi_k(\xi) \phi_k(x) \omega(\xi) \]
where the contour integration is taken in the counterclockwise direction around the infinitely large circle \( C_\infty \). Formally, we know that
\[ \delta(x - \xi) = \sum_{k=1}^{\infty} \phi_k(x) \phi_k(\xi) \omega(\xi) \]
so that,
\[ \frac{1}{2\pi i} \int_{C_\infty} G(x, \xi; \lambda) d\lambda = -\delta(x - \xi), \]
which is the formula we sought.

As an illustration of how to use this formula, consider the equation

$$-\frac{1}{x}(xG')' - \lambda G = \delta(x - \xi)$$

with boundary conditions $G'(0) = 0$, $G(1) = 0$. Recalling that the solutions of the homogeneous equation are the Bessel functions $J_0(\sqrt{\lambda}x)$ and $Y_0(\sqrt{\lambda}x)$, we can write down that

$$G(x, \xi; \lambda) = \begin{cases} \frac{-J_0(\sqrt{\lambda}x)V(\sqrt{\lambda}\xi)}{\sqrt{\lambda}W\left(J_0(\sqrt{\lambda}\xi), V(\sqrt{\lambda}\xi)\right)} & 0 \leq x < \xi \leq 1 \\ \frac{-J_0(\sqrt{\lambda}\xi)V(\sqrt{\lambda}x)}{\sqrt{\lambda}W\left(J_0(\sqrt{\lambda}\xi), V(\sqrt{\lambda}x)\right)} & 0 \leq \xi < x \leq 1 \end{cases}$$

where

$$V(\sqrt{\lambda}x) = Y_0(\sqrt{\lambda})J_0(\sqrt{\lambda}x) - Y_0(\sqrt{\lambda}x)J_0(\sqrt{\lambda}),$$

and $W(u, v) = uv' - uu'$ is the Wronskian of its two arguments.

We must first calculate the Wronskian

$$W\left(J_0(\sqrt{\lambda}x), V(\sqrt{\lambda}x) \right) = -J_0(\sqrt{\lambda}x)W\left(J_0(\sqrt{\lambda}x), Y_0(\sqrt{\lambda}x) \right)$$

Recall from the definition of $Y_\nu(z)$ (Chapter 6) that

$$Y_\nu(z) = \frac{J_\nu(z)\cos \nu\pi - J_{-\nu}(z)}{\sin \nu\pi}$$

so that $W\left(J_\nu(z), Y_\nu(z) \right) = \frac{1}{\sin \nu\pi}W(J_{-\nu}, J_\nu)$. We know that for any two solutions $u, v$ of Bessel's equation, $W(u, v) = k/z$. To calculate the correct value of $k$, we note that near $z = 0$, $J_\nu(z) \sim \left(\frac{z}{2}\right)^\nu \frac{1}{\Gamma(\nu+1)}$ so that

$$W(J_{-\nu}(z), J_\nu(z)) = \frac{2\nu}{z\Gamma(1+\nu)\Gamma(1-\nu)} = \frac{2}{\pi z}\sin \frac{\pi \nu}{2}$$

using an identity for the gamma function (Chapter 6).

It follows that

$$G(x, \xi; \lambda) = \begin{cases} \frac{\pi J_0(\sqrt{\lambda}x)V(\sqrt{\lambda}\xi)\xi}{2J_0(\sqrt{\lambda})}, & 0 \leq x < \xi \leq 1, \\ \frac{\pi J_0(\sqrt{\lambda}\xi)V(\sqrt{\lambda}x)\xi}{2J_0(\sqrt{\lambda})}, & 0 \leq \xi < x \leq 1. \end{cases}$$

To calculate the integral $\frac{1}{2\pi i} \int_{C_\infty} G(x, \xi; \lambda) d\lambda$, we use that there are an infinite number of positive real roots of $J_0(\mu_k) = 0$, so that for $\lambda = \mu_k^2$ the
Green’s function has simple poles. To evaluate the residue at these poles, we note that 
\[ W(\nu; z, \mu; \nu) = \frac{2}{\pi z}, \] 
and if \( \lambda = \mu^2 = \lambda_k \), then 
\[ J_0(\sqrt{\lambda_k})Y_0(\sqrt{\lambda_k}) = \frac{-2}{\pi \sqrt{\lambda_k}, \] 
and \( V(\sqrt{\lambda x}) = Y_0(\sqrt{\lambda_k})J_0(\sqrt{\lambda x}) \). It follows that
\[ \delta(x - \xi) = 2 \sum_{k=1}^{\infty} \frac{J_0(\sqrt{\lambda_k} \xi)J_0(\sqrt{\lambda_k} x)}{[J_0'(\sqrt{\lambda_k})]^2}. \]

This representation of the delta function induces the transform based on Bessel functions
\[ f(x) = \sum_{k=1}^{\infty} \alpha_k J_0(\mu_k x), \]
\[ \alpha_k = \frac{2}{[J_0'(\mu_k)]^2} \int_0^1 f(x)J_0(\mu_k x) dx. \]

The beauty of the formula relating the contour integral of the Green’s function and the delta function is that it gives us an algorithm to generate many different transforms, even those for which the usual eigenfunction expansion techniques do not work. Of course, if we use this formula as an algorithm, it behooves us to come up with an independent proof of the validity of the derived transform pair.

As an example of a transform pair for an operator that does not have eigenfunctions, consider the Green’s function
\[ -G'' - \lambda G = \delta(x - \xi) \]
with \( G(0) = 0 \) and \( G \) in \( L^2[0, \infty) \). If \( \lambda \) is not a positive real number, we calculate directly that
\[ G(x, \xi; \lambda) = \begin{cases} \sin \sqrt{\lambda} xe^{i \sqrt{\lambda} \xi} / \sqrt{\lambda}, & 0 \leq x < \xi < \infty, \\ \sin \sqrt{\lambda} \xi e^{i \sqrt{\lambda} x} / \sqrt{\lambda}, & 0 \leq \xi < x < \infty. \end{cases} \]

Notice that \( e^{i \sqrt{\lambda} x} \) is exponentially decaying as \( x \to \infty \) if \( \text{Im}(\sqrt{\lambda}) > 0 \). We want to evaluate \( \frac{1}{2\pi i} \int_{C_{\infty}} G(x, \xi; \lambda) d\lambda \) around the large circle \( C_{\infty} \) in the counterclockwise direction. However, there is a branch point at \( \lambda = 0 \) and therefore, a branch cut along the positive real axis. Since \( G(x, \xi; \lambda) \) is analytic in \( \lambda \) everywhere else, we can deform \( C_{\infty} \) down to a contour \( C_2 \) which traverses in from \( \text{Re}(\lambda) = \infty \) along the branch cut with \( \text{Im}(\lambda) \) slightly positive and then out to \( \text{Re}(\lambda) = \infty \) along the branch cut with \( \text{Im}(\lambda) \) slightly negative. We make the change of variables \( \lambda = \nu^2 \) so that the branch cut is “unfolded” and our path
of integration becomes the real axis from $\text{Re}\,\mu = \infty$ to $\text{Re}\,\mu = -\infty$. Notice that $\frac{\sin \sqrt{\lambda x}}{\sqrt{\lambda}}$ and $\cos \sqrt{\lambda x}$ are entire functions, and the integral of an entire function around any closed path is zero. As a result,

$$G(x, \xi; \lambda) = i \frac{\sin \sqrt{\lambda x} \sin \sqrt{\lambda \xi}}{\sqrt{\lambda}} + \text{an entire function}$$

for all $x, \xi$. Thus we evaluate

$$\frac{1}{2\pi i} \int_{C_\infty} G(x, \xi; \lambda) d\lambda = \frac{1}{2\pi} \int_{C_2} \frac{\sin \sqrt{\lambda x} \sin \sqrt{\lambda \xi}}{\sqrt{\lambda}} d\lambda$$

$$= -\frac{1}{\pi} \int_{-\infty}^{\infty} \sin \mu x \sin \mu \xi d\mu = -\delta(x - \xi),$$

so that

$$\delta(x - \xi) = \frac{2}{\pi} \int_{0}^{\infty} \sin \mu x \sin \mu \xi d\mu.$$ 

This representation is equivalent to the transform pair

$$\phi(x) = \frac{2}{\pi} \int_{0}^{\infty} \hat{\phi}(\mu) \sin \mu x d\mu$$

$$\hat{\phi}(\mu) = \int_{0}^{\infty} \phi(x) \sin \mu x dx,$$

and is called the FOURIER SINE INTEGRAL TRANSFORM.

We can verify the validity of the Fourier Sine transform directly for any function $\phi$ which is continuously differentiable and has compact support. Consider first the integral

$$\int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} e^{i\mu x} d\mu \right) \phi(x) dx = \lim_{R \to \infty} \int_{-\infty}^{\infty} \left( \int_{-R}^{R} e^{i\mu x} d\mu \right) \phi(x) dx$$

$$= 2\pi \lim_{R \to \infty} \int_{-\infty}^{\infty} \phi(x) \frac{\sin Rx}{\pi x} dx.$$

Recall from Chapter 4 that $S_k(x) = \frac{\sin kx}{\pi x}$ is a delta sequence for Lipschitz continuous functions. As a result,

$$\int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} e^{i\mu x} d\mu \right) \phi(x) dx = 2\pi \phi(0)$$

so that

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\mu x} d\mu = \delta(x)$$
7.2 FOURIER TRANSFORMS

is a representation of the delta function. It follows that

\[ \frac{1}{\pi i} \int_{-\infty}^{\infty} \sin(\mu x) e^{i\mu \xi} d\mu = \frac{-1}{2\pi} \int_{-\infty}^{\infty} \left( e^{i\mu(x+\xi)} - e^{i\mu(x-\xi)} \right) d\mu \]

\[ = \delta(x - \xi) - \delta(x + \xi). \]

Since the domain of the differential operator was defined to be \( L^2[0, \infty) \) with \( u(0) = 0 \), test functions \( \phi(x) \) have support on \( [0, \infty) \) and hence are identically zero for \( x < 0 \). For these test functions \( \delta(x + \xi) \equiv 0 \), and the validity of the transform follows.

We leave it as an exercise for the reader to verify that the FOURIER-COSINE INTEGRAL TRANSFORM

\[ F(\mu) = \int_0^{\infty} f(x) \cos \mu x \, dx \]

\[ f(x) = \frac{2}{\pi} \int_0^{\infty} F(\mu) \cos \mu x \, d\mu \]

arises from the delta function representation for \( Lu = -u'' \) with \( u'(0) = 0 \), \( u \) in \( L^2[0, \infty) \).

The FOURIER INTEGRAL TRANSFORM is one of the most important transform pairs. It can be derived from the operator \( Lu = -u'' \) defined on \( L^2(-\infty, \infty) \) using the Green's function \( G \) defined by \( -G'' - \lambda G = \delta(x - \xi) \). In Chapter 4 we found that this Green's function is given by

\[ G(x, \xi; \lambda) = \frac{-e^{i\sqrt{\lambda}|x-\xi|}}{2i\sqrt{\lambda}}. \]

We evaluate

\[ -\frac{1}{2\pi i} \int_{C_\infty} G(x, \xi; \lambda) d\lambda = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\mu|x-\xi|} d\mu = \delta(x - \xi) \]

which agrees with our earlier calculation. This representation leads to the Fourier Transform Theorem.

**Theorem 7.1**

If \( f(t) \) is piecewise continuously differentiable and is in \( L^1(-\infty, \infty) \), then

\[ f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\mu x} \hat{f}(\mu) d\mu \]

\[ \hat{f}(\mu) = \int_{-\infty}^{\infty} e^{i\mu x} f(x) dx. \]